

# CONNECT FOUR AND GRAPH DECOMPOSITION

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**ABSTRACT.** We introduce the standard decomposition, a way of decomposing a labeled graph into a sum of certain labeled subgraphs. We motivate this graph-theoretic concept by relating it to Connect Four decompositions of standard sets. We prove that all standard decompositions can be generated in polynomial time, which implies that all Connect Four decompositions can be generated in polynomial time.

## 1. INTRODUCTION

Let  $G$  be a directed graph. We say that an integer-valued labeling on the nodes of  $G$  is *compatible with the edge relation* if for all edges  $(a, b)$ , the label of node  $a$  is smaller than the label of node  $b$ . Graphs satisfying that compatibility form the class of *standard graphs*; they are the objects of study of the present paper.

The paper is in two parts. In the first part we study standard graphs and introduce a way of decomposing a standard graph as a sum of *standard components* — these are the standard subgraphs of  $G$  where all labels are 0 or 1. Here addition of labeled graphs is defined as addition of the labels. The main theorem in the first part of the paper is that it is possible to generate all the standard decompositions of a graph in polynomial time.

In the second part of the paper we show that decomposition into standard components is equivalent to a previously studied subject — *Connect Four decomposition* of standard sets  $\Delta \subseteq \mathbb{N}^d$ . Connect Four decomposition is in turn relevant to the study of the Hilbert scheme of points [3], which is what originally prompted the work in this paper.

We conclude our paper with two appendices whose purpose is to provide more evidence for the naturalness of the problems studied here. The first appendix presents a generating function for the number of C4 decompositions of a standard set. The second appendix shows that the set of all standard sets in  $\mathbb{N}^d$  of a given size  $n$  is in canonical bijection with the set of  $(d - 1)$ -fold iterated partitions of  $n$ .

The algorithm in the first part of the paper is based on reducing the problem of computing all standard decompositions of  $G$  to the problem of computing all standard decompositions of  $G$  containing a fixed node  $v$ . We then solve that problem in a recursive way. Any choice of the node  $v$  results in a correct algorithm, yet we give a specific choice of  $v$  that allows the algorithm to generate its output in polynomial time.

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The proof of equivalence between standard decomposition of labeled graphs and Connect Four decomposition of standard sets in the second part of the paper is done in several steps. We first attach a graph  $G(\Delta)$  to each standard set  $\Delta$  such that the standard decompositions of  $G(\Delta)$  and the Connect Four decompositions of  $\Delta$  are in canonical bijection. From  $G(\Delta)$  we then define another graph  $G'(\Delta)$  that is easier to work with called the *canonicalized standard graph*, which also has the same decompositions (see Proposition 26). We show that all labeled graphs arising from standard sets in this way have three specific properties, namely,

- they are standard,
- they are connected, and
- they have a unique node of maximal label.

Let  $\mathcal{S}$  be the class of labeled graphs satisfying these conditions. The connectedness assumption in the definition of  $\mathcal{S}$  is not essential for the complexity of the graphs from that class, since the standard decompositions of a disjoint unions of graphs is the product of the standard decompositions of the individual graphs. We prove in Proposition 30 that each connected graph in  $\mathcal{S}$  arises from a standard set if, in addition, the relation on the nodes of the graph defined by the edges of the graph is transitive. In Proposition 31, we show that for each connected standard graph, there exists a graph in  $\mathcal{S}$  such that the standard decompositions of the two graphs are in canonical bijection. These two propositions will imply that equivalence that we wish to show, namely that the two problems

- (1) computing standard decompositions of labeled graphs and
- (2) computing C4 decompositions of finite standard sets

are equivalent.

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## 2. STANDARD GRAPHS AND STANDARD COMPONENTS

All graphs under consideration have finitely many nodes and are directed — that is, each edge has an orientation. Moreover, we assume that our graphs contain no parallel edges and no loops. The latter conditions means that the edge set induces a partial order on the node set, by setting  $a < b$  if, and only if,  $a$  and  $b$  can be connected by a sequence of edges<sup>1</sup>. In addition to that, all graphs under consideration have integer-valued labelings on the node sets.

**Definition 1** (Labeled graph). A *labeled graph* is a graph  $G$  with a finite node set  $\mathcal{V}_G$ , an edge set  $\mathcal{E}_G \subseteq \mathcal{V}_G \times \mathcal{V}_G$  such that the graph contains no loops, and a labeling of nodes  $\mathcal{L}_G: \mathcal{V}_G \rightarrow \mathbb{Z}$ .

Since the edge set is not a multiset this definition does not allow parallel edges. The constraints on parallel edges and loops are not important to the theory that we present, but we impose those conditions for simplicity since loops and parallel edges add nothing interesting to the problem.

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<sup>1</sup>This partial order is the transitive closure of the relation on the nodes induced by the edges

**Definition 2** (Standard graph). A labeled graph  $G$  is *standard* if all labels are non-negative and the labeling is compatible with the partial order on the nodes in the sense that  $\mathcal{L}_G(a) \leq \mathcal{L}_G(b)$  for all edges  $(a, b) \in \mathcal{E}_G$ .

Note that nodes of standard graphs are allowed to have the label zero. The aim of the present paper is to study the class of standard graphs. To that end, we introduce the operations of *addition* and *subtraction* on labeled graphs.

**Definition 3** (Addition and subtraction). Let  $G$  and  $H$  be labeled graphs. Then  $G \oplus H$  is the labeled graph with node set  $\mathcal{V}_{G \oplus H} := \mathcal{V}_G \cup \mathcal{V}_H$ , edge set  $\mathcal{E}_{G \oplus H} := \mathcal{E}_G \cup \mathcal{E}_H$  and labeling

$$\mathcal{L}_{G \oplus H} := \begin{cases} \mathcal{L}_G & \text{for } v \in \mathcal{V}_G \setminus \mathcal{V}_H, \\ \mathcal{L}_G + \mathcal{L}_H & \text{for } v \in \mathcal{V}_G \cap \mathcal{V}_H, \\ \mathcal{L}_H & \text{for } v \in \mathcal{V}_H \setminus \mathcal{V}_G. \end{cases}$$

We define  $G \ominus H$  to have the same node set and edge set as  $G \oplus H$ , but with labeling

$$\mathcal{L}_{G \ominus H} := \begin{cases} \mathcal{L}_G & \text{for } v \in \mathcal{V}_G \setminus \mathcal{V}_H, \\ \mathcal{L}_G - \mathcal{L}_H & \text{for } v \in \mathcal{V}_G \cap \mathcal{V}_H, \\ -\mathcal{L}_H & \text{for } v \in \mathcal{V}_H \setminus \mathcal{V}_G. \end{cases}$$

The sum of two standard graphs *on the same set of nodes* is again standard. This does not have to be the case for two standard graphs with different sets of nodes.

**Example 4.** Figure 1 shows the sum of two standard graphs, which is not a standard graph.

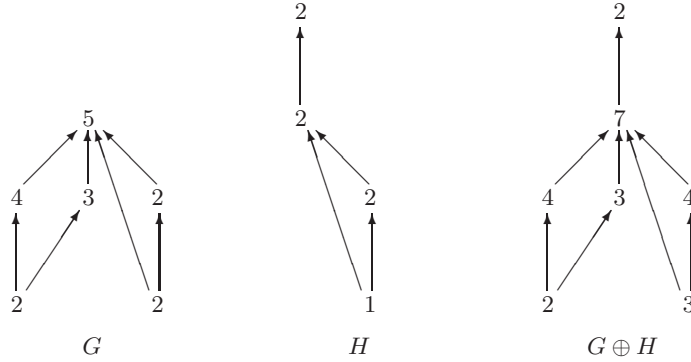


FIGURE 1. The sum of two standard graphs

If we take a standard graph and change the labeling so that all positive labels become 1, then we obtain another standard graph. This is a standard *0-1 graph* — a graph that is a standard graph and a *0-1 graph*.

**Definition 5** (0-1 graph). A labeled graph is a *0-1 graph* if all labels are 0 or 1.

Some subsets  $H$  of a standard graph  $G$  also give rise to standard 0-1 graphs in this way and in some cases we can write  $G$  as  $H + G'$  where  $G'$  is another standard graph. In this case we call  $H$  a *standard component* of  $G$ .

**Definition 6** (Standard component). Let  $G$  and  $H$  be labeled graphs. Then  $H$  is a *standard component* of  $G$  if

- (1)  $H$  is a standard 0-1 graph;
- (2)  $G \ominus H$  is a standard graph; and
- (3) not all labels in  $H$  are zero.

This definition implies that a graph has at least one standard component if, and only if, it is a standard graph. We think of standard components of  $G$  as the building blocks of  $G$ . Our goal is to determine all the ways to build a graph out of such building blocks.

**Definition 7** (Standard decomposition). Let  $G$  be a labeled graph. A multiset of labeled graphs  $\mathcal{H}$  is a *standard decomposition* of  $G$  if each  $H \in \mathcal{H}$  is a standard component of  $G$  and  $G = \sum \mathcal{H}$ . We denote the set of standard decompositions of  $G$  by  $\mathcal{D}(G)$ .

**Example 8.** Figure 2 shows a standard graph and all its decompositions. We do not show the directions of edges in the standard components in order to avoid cluttering the picture.

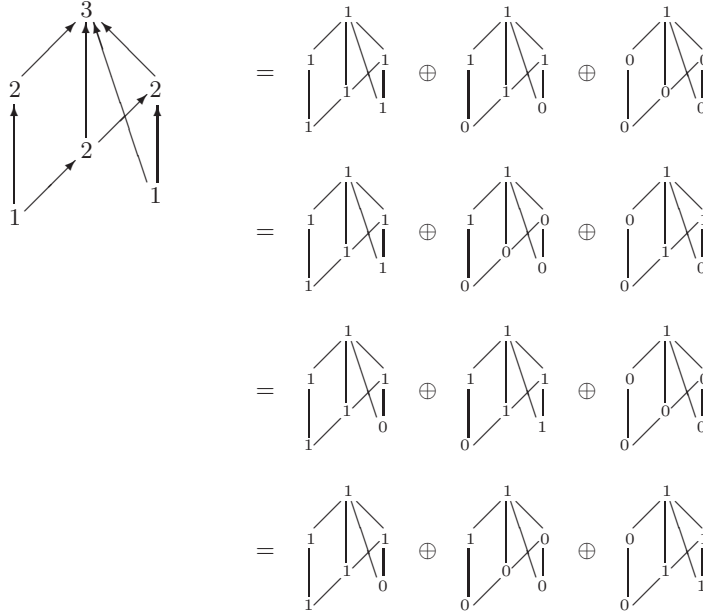


FIGURE 2. All decompositions of a graph.

Observe that a graph admits at least one standard decomposition if, and only if, it is a standard graph. Note that every standard component of a standard graph is a member of at least one standard decomposition. We let standard decompositions be multisets because a standard component can appear multiple times within one decomposition.

We conclude this section with a remark on the class of graphs under consideration. We stated earlier in this section that loops and parallel edges are not interesting in the theory of standard graphs. We now state that

- neither are cycles in  $G$ ,
- nor nodes with label zero,
- nor edges  $(a, b) \in \mathcal{E}_G$  with  $\mathcal{L}_G(a) = \mathcal{L}_G(b)$ .

To support these statements, let  $H$  be a standard component of  $G$  and let  $(a, b)$  be an edge of  $G$  such that  $\mathcal{L}_G(a) = \mathcal{L}_G(b)$ . Then  $\mathcal{L}_H(a) = 1$  if, and only if,  $\mathcal{L}_H(b) = 1$ . So to study standard decompositions, we might as well contract all such edges  $(a, b)$ , merging the nodes  $a$  and  $b$  into a single node. If a standard graph has a cycle, then the labels along the cycle are all equal, so contracting same-label edges removes all cycles. Nodes with label zero are not relevant for standard decomposition either, so we can get rid of those nodes too. In this way we can think of a 0-1 graph as the graph consisting only of the nodes with label 1. Combining these ideas, we get the notion of a *canonical labeled graph*.

**Definition 9** (Canonical labeled graph). A labeled graph  $G$  is *canonical* if

- (1)  $G$  is standard;
- (2)  $G$  has no cycles;
- (3) all labels are positive;
- (4)  $\mathcal{L}_G(a) < \mathcal{L}_G(b)$  for all edges  $(a, b) \in \mathcal{E}_G$ .

We see that given a standard graph  $G$ , there is a canonical graph  $G'$  such that the standard decompositions and standard components of  $G$  and  $G'$  are related by a bijection. We call the process of replacing  $G$  with  $G'$  *canonicalization*. This notion of canonicalization is not to be confused with the usual notion of graph canonicalization, which has to do with isomorphism classes of graphs.

In most of what follows we are not going to need the concept of canonicalization — the theorems, proofs and algorithms work just fine on graphs with cycles, zero labels and equal labels on adjacent nodes. Canonical graphs do, however, make for graphs that are nicer to look at and faster computer programs. We will return to the topic of canonicalization in Proposition 26.

### 3. STANDARD NODE DECOMPOSITIONS

We now turn to the topic of the computational complexity of the problem of computing standard decompositions. We start with a simple illuminating example.

**Example 10.** Let  $G_n$  be the labeled graph defined by

$$\mathcal{V}_{G_n} := \{y, x_1, \dots, x_n\}, \quad \mathcal{E}_{G_n} := \{(x_1, y), \dots, (x_n, y)\},$$

and  $\mathcal{L}_{G_n}(x_i) := 1$  for  $i = 1, \dots, n$ , while  $\mathcal{L}_{G_n}(y) := 2$ . There are  $2^n$  standard components of  $G_n$ , corresponding to the  $n$  independent choices of whether to include or exclude each  $x_i$ . The standard decompositions of  $G_n$  are pairs of standard components that include complementary subsets of  $\{x_1, \dots, x_n\}$ . So  $G_n$  has  $2^{n-1}$  standard decompositions while having only  $n + 1$  nodes.

Consider the computational problem whose input is a labeled graph  $G$  and whose output is the set of standard decompositions  $\mathcal{D}(G)$ , if any. Recall that  $\mathcal{D}(G) \neq \emptyset$  if, and only if,  $G$  is standard — however, we will formulate our statements for arbitrary labeled graphs, thus covering also the case where the output is the empty set.

Example 10 shows that this computation cannot be done in time better than exponential in the worst case since just writing down the output can take exponential time. For problems such as this, it is standard practice to consider an alternative notion of complexity, *generating complexity*, in which we consider the running time as a function of the combined size of the input *and* the output.

We present an algorithm for standard decomposition of graphs that runs in polynomial time in the combined size of input and output. This algorithm is based on the following notion of decomposing a single node of a standard graph.

**Definition 11** (Standard node decomposition). Let  $G$  be a labeled graph and let  $v$  be a node of  $G$ . A multiset of standard graphs  $\mathcal{H}$  is a *standard  $v$ -decomposition* of  $G$  if

- (1) each  $H \in \mathcal{H}$  is a standard component of  $G$ ,
- (2)  $\mathcal{L}_H(v) = 1$  for all  $H \in \mathcal{H}$ ,
- (3)  $G \ominus \sum \mathcal{H}$  is standard,
- (4)  $|\mathcal{H}| = \mathcal{L}_G(v)$ .

We denote the set of standard  $v$ -decompositions of  $G$  by  $\mathcal{D}_v(G)$ .

Consider a standard graph  $G$  with a standard decomposition  $\mathcal{H}$  and a node  $v$  of  $G$ . The subset of  $\mathcal{H}$  whose elements give  $v$  a label of 1 forms a standard  $v$ -decomposition of  $G$ . Another way of characterizing a standard  $v$ -decomposition is that it is a minimal set  $\mathcal{H}$  of standard components of  $G$  such that  $G \ominus \sum \mathcal{H}$  gives  $v$  the label 0 and such that  $\mathcal{H}$  can be extended to a standard decomposition of  $G$ .

Every standard graph has at least one standard decomposition, so Proposition 12 implies that if we can generate standard  $v$ -decompositions in polynomial time, then we can also generate standard decompositions in polynomial time.

**Proposition 12.** Let  $v$  be a node of a labeled graph  $G$ . Then

$$\mathcal{D}(G) = \left\{ \mathcal{H} \cup \mathcal{H}' \mid \mathcal{H} \in \mathcal{D}_v(G), \mathcal{H}' \in \mathcal{D}(G \ominus \sum \mathcal{H}) \right\},$$

where no decomposition appears twice on the right hand side.

*Proof.*  $\subseteq$ : Let  $D \in \mathcal{D}(G)$  and let  $\mathcal{H}$  be the submultiset of  $D$  whose elements give  $v$  the label 1. Then  $\mathcal{H} \in \mathcal{D}_v(G)$ . It only remains to prove that  $D \setminus \mathcal{H} \in \mathcal{D}(G \ominus \sum \mathcal{H})$ , which follows from Lemma 13 below.

$\supseteq$ : Let  $\mathcal{H} \in \mathcal{D}_v(G)$  and let  $\mathcal{H}' \in \mathcal{D}(G \ominus \sum \mathcal{H})$ . Then  $\mathcal{H}' \cup \mathcal{H}$  is a standard decomposition of  $G$  by Lemma 13.

**no duplicates:** Let  $\mathcal{H}, \mathcal{H}'' \in \mathcal{D}_v(G)$  such that  $\mathcal{H} \neq \mathcal{H}''$ . Let  $A \in \mathcal{D}(G \ominus \sum \mathcal{H})$  and  $A'' \in \mathcal{D}(G \ominus \sum \mathcal{H}'')$ . Then  $\mathcal{H} \cup A \neq \mathcal{H}'' \cup A''$  since  $\mathcal{H} \neq \mathcal{H}''$  and  $A \cup A''$  is disjoint from  $\mathcal{H} \cup \mathcal{H}''$ , as the elements of  $A \cup A''$  give  $v$  the label zero, while the elements of  $\mathcal{H} \cup \mathcal{H}''$  give  $v$  the label 1.  $\square$

**Lemma 13.** Let  $G$  be a labeled graph. Let  $A$  be a multiset of standard 0-1 subgraphs of  $G$  and let  $B$  be a submultiset of  $A$ . Then  $A$  is a standard decomposition of  $G$  if, and only if,  $A \setminus B$  is a standard decomposition of  $G \ominus \sum B$ .

*Proof.* **if:** Assume that  $A \setminus B$  is a standard decomposition of  $G \ominus \sum B$ . Then  $G \ominus \sum B = \sum(A \setminus B)$  so  $G = \sum A$ . It only remains to prove that each  $a \in A$  is a standard component of  $G$ . To prove that, we need to show that  $G \ominus a$  is standard. We already know that  $a$  is a standard component of  $G \ominus \sum B$ , which implies that

$G \ominus \sum B \ominus a$  is standard. Then  $G \ominus a = (G \ominus \sum B \ominus a) \oplus \sum B$  is standard, as it is a sum of standard graphs with identical node sets.

**only if:** Assume that  $A$  is a standard decomposition of  $G$ . Then  $G = \sum A$  so  $G \ominus \sum B = \sum(A \setminus B)$ . It only remains to prove that each  $a \in A \setminus B$  is a standard component of  $G \ominus \sum B$ . To prove that we need to show that  $G \ominus \sum B \ominus a$  is standard. We already know that  $G \ominus \sum A$  has all labels zero, so it is standard. Then  $G \ominus \sum B \ominus a = (G \ominus \sum A) \oplus \sum(A \setminus (B \cup \{a\}))$ , so  $G \ominus \sum B \ominus a$  is standard, as it is a sum of standard graphs with identical node sets.  $\square$

#### 4. GENERATING STANDARD NODE DECOMPOSITIONS

Proposition 12 reduces the problem of generating  $\mathcal{D}(G)$  in polynomial time to the problem of generating the standard node decomposition  $\mathcal{D}_v(G)$  in polynomial time for some freely chosen node  $v$  of  $G$ . In this section we investigate this problem. Our solution is based on choosing the right node  $v$  to decompose.

Consider the set of all standard components of  $G$  that give  $v$  the label 1. We impose an ordering,  $H_1, \dots, H_k$ , on the elements of that set. This ordering can be chosen arbitrarily, but is fixed once and for all. Now let  $F$  be any labeled subgraph of  $G$ . For each such  $F$  and each  $i = 1, \dots, k$ , we define

$$\tau(F, i) := \{\mathcal{H} \subseteq \{H_1, \dots, H_i\} \mid \mathcal{H} \text{ is a multiset appearing in } \mathcal{D}_v(F)\}$$

If we can compute  $\tau(F, i)$  in general then we can also compute  $\mathcal{D}_v(G)$  since  $\tau(G, k) = \mathcal{D}_v(G)$ . In order to compute  $\tau(F, i)$ , consider the recursive formula

$$(1) \quad \tau(F, i) = \begin{cases} \{\emptyset\} & \text{if all labels of } F \text{ are zero, else} \\ \emptyset & \text{if } F \text{ is not standard or } i = 0, \text{ else} \\ \tau(F, i-1) \cup \{\mathcal{H} \cup \{H_i\} \mid \mathcal{H} \in \tau(F \ominus H_i, i)\}. \end{cases}$$

This way of writing  $\tau$  immediately suggests an algorithm based on recursively evaluating the expression. It is a problem with this approach that this algorithm can spend a large amount of computational steps to determine that  $\tau(F, i)$  is empty. This is an obstacle to proving that this algorithm generates its output in polynomial time.

We say that a pair  $(F, i)$  is *relevant* if  $\tau(F, i) \neq \emptyset$ , and *irrelevant* otherwise. In these terms, the problem with the algorithm is that it can spend a lot of time on an irrelevant pair before determining that it is irrelevant. For making the algorithm generate its output in polynomial time, we need a criterion for detecting irrelevant pairs. We can use such a criterion to quickly eliminate irrelevant pairs in the algorithm.

The notion of a *maximal standard components* is important for detecting irrelevant pairs.

**Definition 14** (Maximal standard component). The *maximal standard component* of a standard graph  $G$  is the unique standard component  $H$  for which  $\mathcal{L}_H(v) = 1$  if, and only if,  $\mathcal{L}_G(v) > 0$ .

In other words, we obtain the maximal standard component  $H$  by removing all nodes of  $G$  with label zero and giving all remaining nodes a label of 1. This  $H$  is maximal in the sense that it contains all other standard components. Note that the maximal standard component is always a standard component unless we are in the degenerate case where all nodes of  $G$  are labeled zero.

**Proposition 15.** Let  $\mathcal{H}$  be a multiset of standard components of a labeled graph  $G$ . Then  $\mathcal{H}$  can be extended to a standard decomposition of  $G$  if, and only if,  $G \ominus \sum \mathcal{H}$  is standard.

*Proof. if:* If  $G = \sum \mathcal{H}$  then we are done, so suppose that  $G \neq \sum \mathcal{H}$ . Let  $C$  be the maximal standard component of  $G \ominus \sum \mathcal{H}$ . Then  $G \ominus \sum(\mathcal{H} \cup \{C\})$  is standard. The assertion follows from this by induction.

*only if:* If  $\mathcal{H}'$  is a multiset of standard components of  $G$  that contains  $\mathcal{H}$ , and  $G \ominus \sum \mathcal{H}$  is not standard, then neither is  $G \ominus \sum \mathcal{H}'$ , so  $\mathcal{H}'$  is not a standard decomposition of  $G$ .  $\square$

**Corollary 16.** If  $H$  is a standard component of a standard graph  $G$ , then  $H$  is an element of at least one standard decomposition of  $G$ .

The proof of Proposition 15 uses maximal standard components to extend a partial standard decomposition to a full standard decomposition. Proposition 17 expands on this idea.

**Proposition 17.** Let  $v$  be a node of minimal positive label in a labeled graph  $G$ . Let  $\mathcal{H}$  be a multiset of standard components of  $G$  that give  $v$  the label 1. Let  $H$  be the maximal standard component of  $G$ . Assume that  $G \ominus \sum \mathcal{H}$  is standard. Let  $\mathcal{H}'$  be the union of  $\mathcal{H}$  and the multiset containing  $\mathcal{L}_G(v) - |\mathcal{H}|$  copies of  $H$ . Then  $\mathcal{H}'$  is a standard  $v$ -decomposition of  $G$ .

*Proof.* Upon applying the proof of Proposition 15 to  $\mathcal{H}$ , we obtain a standard decomposition  $\mathcal{H}'' \supseteq \mathcal{H}$  of  $G$ . Since the label of  $v$  is minimal among all positive labels appearing in  $G$ , the first  $\mathcal{L}_G(v) - |\mathcal{H}|$  rounds of the inductive construction in that proof will use the same maximal standard component  $H$ . After that the label of  $v$  has become zero, so the maximal standard components used in later rounds of the construction will give  $v$  the label zero. So the subset of  $\mathcal{H}''$  that gives  $v$  the label 1 is precisely  $\mathcal{H}'$ , which implies that  $\mathcal{H}'$  is a standard  $v$ -decomposition of  $G$ .  $\square$

Through choosing wisely the node  $v$  and the order of the standard components  $H_1, \dots, H_k$ , Proposition 18 gives an if-and-only-if criterion for detecting irrelevant pairs.

**Proposition 18.** Given a labeled graph  $G$ , choose  $v$  to be a node of minimal positive label, and choose an order of the standard components  $H_1, \dots, H_k$  giving  $v$  the label 1 such that  $H_1$  is the maximal standard component of  $G$ . Then a pair  $(F, i)$  with  $1 \leq i \leq k$  is relevant if, and only if,  $F$  is standard.

*Proof.* This is a direct consequence of Proposition 17.  $\square$

## 5. GENERATING STANDARD DECOMPOSITIONS IN POLYNOMIAL TIME

Based on the previous two sections, we can now present an algorithm for generating standard decompositions and prove that it runs in polynomial time.

**Theorem 19.** The algorithm in Figure 3 generates the standard decompositions of a labeled graph in polynomial time.

The pseudo code for `standardDecompositions` implements the recursive formula from Proposition 12. The pseudo code for `standardNodeDecompositions`



```

1: function STANDARDDECOMPOSITIONS( $G$ )
2:   if all labels of all nodes of  $G$  are zero then
3:     return  $\{\emptyset\}$ 
4:   else
5:     choose a node  $v \in \mathcal{V}_G$  of minimal positive label
6:      $D \leftarrow \text{STANDARDNODEDECOMPOSITIONS}(G, v)$ 
7:     return  $\{\mathcal{H} \cup \mathcal{H}' \mid \mathcal{H} \in D, \mathcal{H}' \in \text{STANDARDDECOMPOSITIONS}(G \ominus \sum \mathcal{H})\}$ 

8:   end if
9: end function
10: function STANDARDNODEDECOMPOSITIONS( $G, v$ )
11:    $H_1 \leftarrow$  the maximal standard component of  $G$ 
12:    $H_2, \dots, H_k \leftarrow$  all other standard components of  $G$  that give  $v$  the label 1
13:    $S \leftarrow \{H_1, \dots, H_k\}$ 
14:   return  $\text{TAU}(G, k, S)$ 
15: end function
16: function  $\text{TAU}(F, i, S)$ 
17:   if all labels of  $F$  are zero then
18:     return  $\{\emptyset\}$ 
19:   else
20:     if  $F$  is not a standard graph then
21:       return  $\emptyset$ 
22:     else
23:       return  $\text{TAU}(F, i-1, S) \cup \{\mathcal{H} \cup \{H_i\} \mid \mathcal{H} \in \text{TAU}(F \ominus H_i, i, S)\}$ 
24:     end if
25:   end if
26: end function

```

FIGURE 3. An algorithm for standard decomposition.

implements the recursion from Proposition 4 where the function **Tau** is  $\tau$  from that section. Line 20 eliminates pairs that are irrelevant according to Proposition 18.

In reading the pseudo code for **Tau**, note that the first return is of the value  $\{\emptyset\}$  while the second is of the value  $\emptyset$ . Here  $\{\emptyset\}$  is a set containing one decomposition while  $\emptyset$  is a set containing nothing.

*Proof of Theorem 19.* Recall that *generating output in polynomial time* means that the algorithm runs in polynomial time in the combined size of input and output — this is the meaning of the word “generate” in this context.

The size of the input and output depend on the representation used. We specify a graph as a list of nodes with labels and a list of edges. We specify the set of decompositions as a list of standard components followed by a list of sets that specify a decomposition by referring back to the list of components. Each standard component is specified by a bit per node indicating whether that node is an element of the standard component.

We assume a model where all labels and indices take up one word of space, rather than the logarithmic number of bits actually necessary to hold these numbers. The

only arithmetic operations we perform is subtractions  $a - b$  where  $a > b$  so this assumption does not weaken the theorem.

**standardNodeDecompositions is correct:** Suppose that we call the function `standardNodeDecompositions` on the pair  $(G, v)$ . We know that  $v$  is a node of minimal positive label in  $G$  since `standardDecompositions` always makes calls to `standardNodeDecompositions` with such a  $v$ . Also observe that the sequence  $H_1, \dots, H_n$  are ordered to satisfy the precondition of Proposition 20. We then see that `standardNodeDecompositions` computes the correct value  $\mathcal{D}_v(G)$  since it directly implements the recursive formula from equation 1 along with the criterion for irrelevant pairs from Proposition 18.

**standardNodeDecompositions is polynomial:** Let  $G$  have  $n$  nodes and  $e$  edges. We do not give pseudo code for generating  $H_1, \dots, H_k$ , but it is not difficult to do this in time  $O(k(n + e))$  using backtracking. We first need to prove that  $k(n + e)$  is polynomial in the size of the output.

Let  $l$  be the label of  $v$  in  $G$ . Every  $H_i$  is an element of at least one standard decomposition of  $G$  by Corollary 16, and each  $v$ -decomposition has exactly  $l$  elements, so  $k \leq ld$  where  $d$  is the number of standard  $v$ -decompositions of  $G$ . So computing  $H_1, \dots, H_k$  can be done in time  $O(ld(n + e))$ . The size of the input is  $\Theta(n + e)$  and the size of the output is  $\Theta(ld + kn)$  since it takes  $l$  elements of  $S$  to specify each of the  $d$  decompositions and for each irreducible decomposition we need one bit per node to specify whether it is in the graph or not. Clearly  $ld(n + e) = \Omega(ldn^2)$  is bounded above by a polynomial in  $ld + kn$ , so the time to compute  $S$  is polynomial.

It remains to prove that `Tau` takes polynomial time. Each individual call to `Tau`, not counting recursive calls, can be done in time  $O(n + e)$ . We need an upper bound for the number of recursive calls.

Consider a tree  $T$  where each recursive call to `Tau` is a node labeled by the parameters  $(F, i)$  and where there is an edge from the caller to the callee. The relevant leaves of  $T$  give rise to one distinct node decomposition per leaf so  $d$ , the number of  $v$ -decompositions of  $G$ , is also the number of relevant leaves of  $T$ . Let  $r$  be the number of irrelevant leaves of  $T$  — these do not give rise to a  $v$ -decomposition. Since  $T$  is a binary tree we see that there are  $r + d - 1$  internal nodes in  $T$ . We need an upper bound for  $r$ .

Since Proposition 18 is an if-and-only-if criterion for irrelevant pairs, we see that the sub-tree rooted at any internal node contains a relevant pair. This implies that the sibling of an irrelevant leaf  $A$  is a root of a sub-tree that contains some relevant leaf  $B$ . Let  $f$  be the mapping  $A \mapsto B$ . If  $f(A) = B$  then the parent of  $A$  is on the path from the root of  $T$  to  $B$ . All the relevant leaves are at depth  $k$  or less, so  $f$  can map at most  $k$  irrelevant leaves to each relevant leaf. This implies that  $r \leq dk$ .

We have seen that there are  $d$  relevant leaves, at most  $dk$  irrelevant leaves and therefore also at most  $d + dk$  internal nodes in  $T$ , which is a total of at most  $2d + 2dk$  nodes. So the time taken by all recursive calls to `Tau` is  $\Omega(dk(n + e))$ . Recall that the input size is  $\Theta(n + e)$  and the output size is  $\Theta(ld + kn)$ . Clearly  $dk(n + e)$  is dominated by a polynomial in  $(n + e) + (ld + kn)$ . This proves that `standardNodeDecompositions` generates  $\mathcal{D}_v(G)$  in polynomial time.

**standardDecompositions is correct:** We have already done the correctness proof since `standardDecompositions` directly implements the recursive formula for  $\mathcal{D}(G)$  from Proposition 12.

**standardDecompositions is polynomial:** We have already seen that each call to `standardNodeDecompositions` generates its own output in polynomial time. Consider as before a tree  $T$  where each recursive call to `standardDecompositions` is a node with an edge from the caller to the callee. Let  $q$  be the number of leaves of  $T$ . Every leaf contributes at least one distinct decomposition to the output, so  $q$  is a lower bound on the number of decompositions of  $G$ . The multiset of node decompositions computed by all the calls to `standardNodeDecompositions` is in bijection with the edges of  $T$ . All trees have more nodes than edges and more leaves than internal nodes so the combined time to compute all the node decompositions is dominated by a polynomial in  $q(n + e)$  where  $n + e$  is the input size for the original input which is an upper bound on the size of any graph produced during the computation.

Line 7 could a priori cause a huge workload in going through all the elements of  $D$ . However, we can charge this work to each of the children of that node that are produced in this way which clears up the problem. As trees have more leaves than internal nodes the total number of nodes of  $T$  is less than  $2q$ . This proves that the total time to compute  $\mathcal{D}(G)$  is bounded by a polynomial in  $w(n + e)$  where  $w$  is the number of decompositions and  $\Theta(n + e)$  is the size of the input.  $\square$

We can extract some bounds on the number of node decompositions from the arguments just given.

**Proposition 20.** Let  $v$  be a node of a standard graph  $G$ . Let  $l := \mathcal{L}_G(v)$ . If  $G$  has  $k$  standard components that give  $v$  label 1, then there are between  $\frac{k}{l}$  and  $\binom{k}{l}$  standard  $v$ -decompositions of  $G$ . If  $v$  is a node of minimal positive label in  $G$ , then there are at least  $k$  standard  $v$ -decompositions of  $G$ .

*Proof.* Every  $v$ -decomposition of  $G$  has exactly  $l$  elements, and the elements are chosen among the  $k$  standard components that give  $v$  the label 1, so there cannot be more than  $\binom{k}{l}$  standard  $v$ -decompositions.

Every one of the  $k$  standard components can be extended to a standard decomposition of  $G$  by Corollary 16 and therefore also to a standard  $v$ -decomposition. We get the minimal number of standard  $v$ -decompositions when each of these extensions are unique. As each standard  $v$ -decomposition has  $l$  elements, that implies the existence of at least  $\frac{k}{l}$  standard  $v$ -decompositions.

If  $v$  is a label of minimal positive label, then each standard component  $H$  that gives  $v$  the label 1 can be extended to a  $v$ -decomposition using only the maximal standard component by Proposition 17. So there are at least  $k$  standard  $v$ -decompositions in this case.  $\square$

## 6. FROM STANDARD SETS TO STANDARD GRAPHS

In the remaining three sections we closely investigate the relation between standard decomposition of labeled graphs and another combinatorial problem called *Connect Four decomposition*. Connect Four decomposition will probably at first appear to have no obvious connection to standard decomposition and once the connection is clear it may appear to be only a specific case of standard decomposition. In the end we show that the two problems are equivalent.

We initially considered Connect Four decompositions as part of a research project on the Hilbert scheme of points [3] and the theory of standard decomposition grew out of that work. Even so, we have chosen to present Connect Four decomposition

after standard decomposition in this paper because standard decomposition is the easier concept to define and understand.

A *standard set*, or *staircase*, is a subset  $\Delta \subseteq \mathbb{N}^d$  whose complement  $C := \mathbb{N}^d \setminus \Delta$  satisfies  $C + \mathbb{N}^d = C$ . We are only going to consider standard sets of finite cardinality. Standard sets in  $\mathbb{N}$  are just intervals starting at 0; in  $\mathbb{N}^2$ , they can be identified with partitions, or with Young diagrams<sup>2</sup>; in  $\mathbb{N}^3$ , they are also known as *plane partitions*; in  $\mathbb{N}^d$  for  $d > 3$ , they are also known as *solid partitions*. Standard sets in  $\mathbb{N}^d$  canonically correspond to monomial ideals in the polynomial ring  $k[x_1, \dots, x_d]$ . See Figure 4 for examples in dimensions 1, 2, and 3.

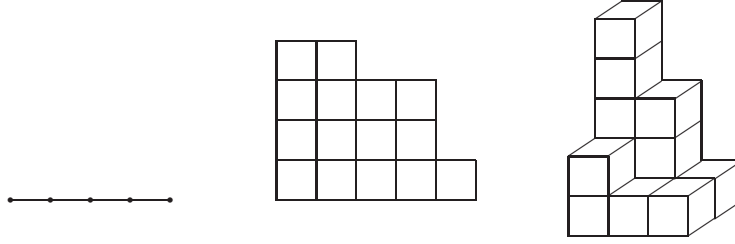


FIGURE 4. Standard sets in dimensions 1, 2 and 3

Consider the projection to the first  $d - 1$  components,  $q^d : \mathbb{N}^d \rightarrow \mathbb{N}^{d-1} : \beta \mapsto (\beta_1, \dots, \beta_{d-1})$  and its complementary projection,  $q_d : \mathbb{N}^d \rightarrow \mathbb{N} : (\beta_1, \dots, \beta_d) \mapsto \beta_d$ . For each standard set  $\Delta$ , we have the equality

$$\Delta = \{ \beta \in \mathbb{N}^d \mid q_d(\beta) < |(q^d)^{-1}(q^d(\beta)) \cap \Delta| \}.$$

The integer  $|(q^d)^{-1}(q^d(\beta)) \cap \Delta|$  appearing on the right-hand side is the cardinality of the fiber of the projection  $q^n : \Delta \rightarrow \mathbb{N}^{d-1}$  over the point  $\gamma := q^d(\beta)$ . We call that quantity the *height* of  $\Delta$  over  $\gamma$ . The equation displayed above implies that the datum of standard set  $\Delta$  is equivalent to the datum of the projection  $\Delta' := q^d(\Delta)$ , which is a standard set in  $\mathbb{N}^{d-1}$ , and the datum of the heights over all  $\gamma \in \Delta'$ . The heights satisfy a compatibility condition: Upon denoting by  $h_\gamma$  the height over  $\gamma \in \Delta'$ , we see that  $h_{\gamma+e_i} \leq h_\gamma$  for all standard basis elements  $e_i \in \mathbb{N}^{d-1}$  and all  $\gamma \in \Delta'$  such that also  $\gamma + e_i \in \Delta'$ . These observations motivate the following definition:

**Definition 21** (Standard graph of a standard set). Let  $\Delta \subseteq \mathbb{N}^d$  be a finite standard set. We define the *standard graph* of  $\Delta$ , denoted by  $G(\Delta)$ , by setting

$$\begin{aligned} \mathcal{V}_{G(\Delta)} &:= q^d(\Delta), \\ \mathcal{E}_{G(\Delta)} &:= \{ (\gamma', \gamma) \mid \gamma' = \gamma + e_i \text{ for some } i \} \\ \mathcal{L}_{G(\Delta)}(\gamma) &:= |(q^d)^{-1}(\gamma) \cap \Delta|. \end{aligned}$$

<sup>2</sup>in the French notation

The discussion leading to the definition proves that  $G(\Delta)$  is indeed a standard graph. The transition from a standard set to its standard graph is illustrated in the first two pictures in Figure 6.

Addition of standard graphs has a counterpart on standard sets, called *C4 addition*.

**Definition 22** (C4 sum). Let  $\Delta_1$  and  $\Delta_2$  be two finite standard sets in  $\mathbb{N}^{d-1}$ . We define the *Connect Four sum*, or *C4 sum* of  $\Delta_1$  and  $\Delta_2$  by

$$\Delta_1 + \Delta_2 := \left\{ \beta \in \mathbb{N}^n \mid \begin{array}{l} q_d(\beta) < |(q^d)^{-1}(q^d(\beta)) \cap \Delta_1| \\ + |(q^d)^{-1}(q^d(\beta)) \cap \Delta_2| \end{array} \right\}.$$

So for determining the C4 sum of  $\Delta_1$  and  $\Delta_2$ , we define  $\Delta'$  to be the union of  $q^n(\Delta_1)$  and  $q^n(\Delta_2)$  and, for all  $\gamma \in \Delta'$ ,  $h_\gamma$  to be the sum of the heights over  $\gamma$  of  $\Delta_1$  and  $\Delta_2$ <sup>3</sup>. Then  $\Delta$  is characterized by its projection  $\Delta'$  and the heights  $h_\gamma$ .

Here is a more graphic way of thinking about the C4 sum: Place  $\Delta_1$  and  $\Delta_2$  somewhere on the  $d$ -axis in  $\mathbb{N}^d$  such that they do not intersect, subsequently drop the cubes along the  $d$ -axis, until they get stacked above each other on the  $1, 2, \dots, (d-1)$ -hyperplane. The result is the standard set  $\Delta_1 + \Delta_2$ . Figure 5 illustrates that process in two examples. The figure also explains the analogy to the eponymous game *Connect Four*.

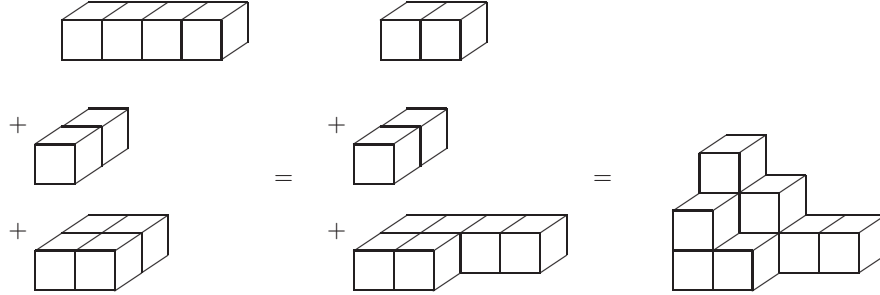


FIGURE 5. C4 sums of 2-dimensional standard sets yielding a 3-dimensional standard set

It is easy to see that

- $\Delta_1 + \Delta_2$  is a standard set;
- its cardinality is the sum of the cardinalities of  $\Delta_1$  and  $\Delta_2$ ;
- C4 addition is associative and commutative, and  $\emptyset$  is its neutral element;
- $G(\Delta_1 + \Delta_2) = G(\Delta_1) \oplus G(\Delta_2)$ .

The last item confirms that C4 addition of standard set is indeed the counterpart of addition of standard graphs. Here is the counterpart of standard decomposition of standard graphs.

<sup>3</sup>We say that the height of  $\Delta_i$  over  $\gamma$  is zero if  $\gamma \notin q^n(\Delta_i)$

**Definition 23** (C4 decomposition). Let  $\Delta \subseteq \mathbb{N}^d$  be a finite standard set. A *C4 decomposition* of  $\Delta$  is a multiset  $\{\Delta_1, \dots, \Delta_h\}$  of standard sets in  $\mathbb{N}^{d-1}$  whose C4 sum equals  $\Delta$ . Here we understand each  $\Delta_i$  to be a standard set in  $\mathbb{N}^d$  via the embedding  $\mathbb{N}^{d-1} \hookrightarrow \mathbb{N}^d : \gamma \mapsto (\gamma, 0)$ .

Figure 5 shows C4 decompositions of the standard set in  $\mathbb{N}^3$  on the right hand side into two (multi)sets of standard set in  $\mathbb{N}^2$ . Note, however, that the three-dimensional standard set of that example has more C4 decompositions than the two shown in the figure.

C4 decompositions naturally arise in the study of lexicographic Gröbner bases of finite sets of points in affine space, see [2], [3] and references therein. We shall explain in Appendix B that C4 decompositions also arise as the combinatorial objects which parametrize iterated partitions of integers.

The following proposition is the first step of three in proving that C4 decomposition and standard decomposition of labeled graphs are equivalent.

**Proposition 24.** Let  $\Delta \subseteq \mathbb{N}^d$  be a finite standard set. Then the C4 decompositions of  $\Delta$  and the standard decompositions of  $G(\Delta)$  are in canonical bijection.

*Proof.* Let  $\{\Delta_1, \dots, \Delta_h\}$  be a C4 decomposition of  $\Delta$ . Consider, for  $j = 1, \dots, h$ , the graph  $H_j$  whose nodes and edges are identical to the nodes and edges of  $G(\Delta)$  and whose labeling is given by

$$\mathcal{L}_{H_j}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in H_j \\ 0 & \text{else.} \end{cases}$$

In other words, we think of  $\Delta_j$ , which is a priori a standard set in  $\mathbb{N}^{d-1}$ , as being a standard set in  $\mathbb{N}^d$ , as we do in Definition 23, and define  $H_j := G(\Delta_j)$ . Then  $H_j$  is obviously a standard 0-1 graph. The fact that  $\{\Delta_1, \dots, \Delta_h\}$  is a C4 decomposition of  $\Delta$  implies that  $\mathcal{H} := \{H_1, \dots, H_h\}$  is a standard decomposition of  $G(\Delta)$ .

Conversely, let  $\mathcal{H}$  be a standard decomposition of  $G(\Delta)$ . Recall that the node set of  $G(\Delta)$  is  $\Delta' := q^d(\Delta)$ , which is a standard set in  $\mathbb{N}^{d-1}$ . For every  $H \in \mathcal{H}$ , we define  $\Delta(H)$  to be the set of all  $\gamma \in \Delta'$  with  $\mathcal{L}_H(\gamma) = 1$ . The definition of  $\mathcal{E}_{G(\Delta)}$ , together with the fact that  $H$  is a standard graph, shows that  $\Delta(H) \subseteq \mathbb{N}^{d-1}$  is a standard set contained in  $\Delta'$ . The fact that  $\mathcal{H}$  is a standard decomposition of  $G(\Delta)$  means that for each  $\gamma \in \Delta'$ , the labels of all nodes  $\gamma$ , which are 0 or 1, sum up to the height  $h_\gamma$ . This means that C4 sum of the corresponding multiset  $\{\Delta(H) \mid H \in \mathcal{H}\}$  equals  $\Delta$ , so that multiset is a C4 decomposition of  $\Delta$ .

The two constructions are readily seen to be mutual inverses.  $\square$

The graph of a given standard set will in general contain many nodes of identical label which are connected by an edge. From the discussion at the end of Section 2, we know that edges between nodes of the same label are irrelevant for computing the standard decomposition of that graph. We also know that we can get rid of those redundancies, without spoiling standard decompositions, by passing from a graph to its canonicalization. Given a standard set  $\Delta$ , we should therefore not work with its standard graph, but rather the canonicalization of its standard graph.

**Definition 25** (Canonicalization of the graph of  $\Delta$ ).

- We say that a subset  $B$  of  $\mathbb{N}^{d-1}$  is *connected* if for all  $\gamma, \gamma' \in B$ , there exists a sequence  $(\gamma_j)$  in  $B$  starting at  $\gamma_0 = \gamma$  and ending at  $\gamma_n = \gamma'$  such

that for all  $j$ , we either have  $\gamma_{j+1} = \gamma_j + e_i$  or  $\gamma_j = \gamma_{j+1} + e_i$  for some  $i \in \{1, \dots, d-1\}$ .

- A *connected component* of  $A \subseteq \mathbb{N}^{d-1}$  is a connected  $B \subseteq A$ , maximal with respect to inclusion.
- Let  $\Delta \subseteq \mathbb{N}^d$  be a standard set,  $h := q_d(\Delta)$  its height, and  $\Delta' := q^d(\Delta)$  its projection. For  $a = 1, \dots, h$ , we define the  $a$ -th *isohypse* as

$$\Delta^a := \{\gamma \in \Delta' \mid |(q^d)^{-1}(\gamma) \cap \Delta| = a\},$$

the set of all points in the projection of height  $a$ .

- We define the graph  $G'(\Delta)$  by

$$\mathcal{V}_{G'(\Delta)} := \{\text{connected components of } \Delta^a \mid a = 1, \dots, h\};$$

$$\mathcal{E}_{G'(\Delta)} := \{(\Delta_b^a, \Delta_d^a) \mid \exists \gamma' \in \Delta_b^a, \gamma \in \Delta_d^a : \gamma' = \gamma + e_i \text{ for some } i\}$$

$$\mathcal{L}_{G'(\Delta)}(\Delta_b^a) := a,$$

where we denote by  $\Delta_b^a$  the connected components of isohypse  $\Delta^a$ .

The transition from  $\Delta$  to  $G(\Delta)$  and to  $G'(\Delta)$  is illustrated in Figure 6.

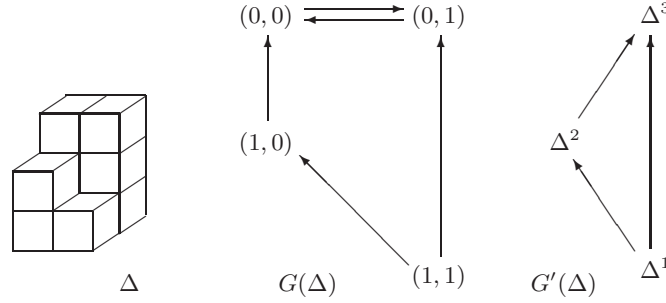


FIGURE 6. A standard set of height 3, its graph, and its canonicalized graph

The following proposition is the second step of three in proving that C4 decomposition and standard decomposition of labeled graphs are equivalent.

**Proposition 26.** Let  $\Delta \subseteq \mathbb{N}^d$  be a finite standard set. Then  $G'(\Delta)$ , as defined above, is the canonicalization of the standard set of  $\Delta$ .

*Proof.* Let  $G(\Delta)$  be the standard graph of  $\Delta$ . Then  $\Delta'$  is the node set of  $G(\Delta)$ , and two nodes get the same label if, and only if, they lie in the same isohypse  $\Delta^a$ . Moreover, the definition of  $G(\Delta)$  shows that two nodes of this graph lying in the same isohypse are connected by a sequence of edges in that graph if, and only if, they lie in the same connected component of some  $\Delta^a$ . So we may contract each connected component  $\Delta_b^a$  of each isohypse to one node. This is what the definition of  $G'(\Delta)$  does.

For finishing the proof, we have to show that no more pairs of nodes in  $G'(\Delta)$  may be contracted into one node. Contraction only happens if two nodes have the same label and are connected by an edge. Suppose that  $\Delta_b^a$  and  $\Delta_d^a$  are connected

by an edge. Then there exist  $\gamma' \in \Delta_b^a$  and  $\gamma \in \Delta_d^a$  such that  $\gamma' = \gamma + e_i$ , so  $\gamma'$  and  $\gamma$  lie in the same connected component of  $\Delta^a$ , a contradiction.  $\square$

#### 7. FROM STANDARD GRAPHS WITH UNIQUE MAXIMAL NODES TO STANDARD SETS

For each standard set  $\Delta$ , the canonicalized graph  $G'(\Delta)$  is connected and contains a unique node of maximal label, namely, the highest isohypse  $\Delta^h$ . This graph thus lies in the class  $\mathcal{S}$  defined in the Introduction. Examples 27 and 28 show that graphs in  $\mathcal{S}$  may or may not arise from standard sets.

**Example 27.** Figure 7 shows a standard graph which arises as the standard graph of a standard set in  $\mathbb{N}^4$ , namely,

$$\Delta = \left\{ \begin{array}{l} (0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 0, 2), \\ (1, 0, 0, 0), (1, 0, 0, 1), (0, 1, 0, 0), (0, 1, 0, 1), \\ (0, 0, 1, 0), (0, 0, 1, 1), (0, 1, 1, 0), (0, 1, 1, 1), \\ (1, 1, 0, 0), (1, 0, 1, 0) \end{array} \right\}.$$

The picture on the right hand side of that figure shows  $\Delta^3, \Delta^2$  and  $\Delta^1 \subseteq \mathbb{N}^3$ .

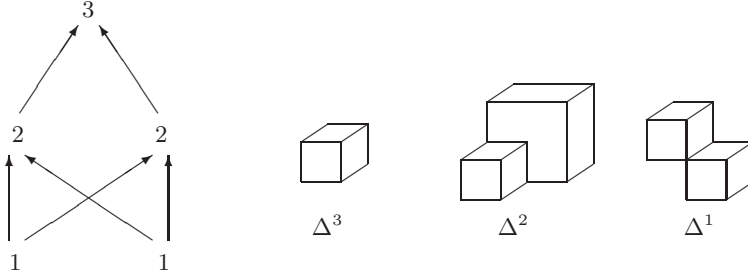


FIGURE 7. A standard graph arising from a standard set in  $\mathbb{N}^4$

**Example 28.** The graph shown in Figure 8 does not arise as the standard graph of a standard set.



FIGURE 8. A graph not arising from a standard set



*Proof.* Assume that  $\Delta \subseteq \mathbb{N}^d$  is a standard set whose standard graph is the given graph  $G$ . In particular, the nodes of  $G$  are the isohypses  $\Delta^i$ , for  $i = 1, 2, 3, 4$ . We claim that there exists an element  $\beta \in \Delta^1$  and  $i, j \in \{1, \dots, d-1\}$  such that  $\beta - e_i \in \Delta^2$  and  $\beta - e_j \in \Delta^4$ . This will finish the proof, since  $\beta - e_i - e_j$  will then lie in  $\Delta$ . But  $\beta - e_i - e_j$  can lie in neither  $\Delta^1$  nor  $\Delta^2$  nor  $\Delta^3$ , since either of these inclusions would contradict the standard set property of  $\Delta$ . However, an inclusion  $\beta - e_i - e_j \in \Delta^4$  would force an edge from 2 to 4 in the graph of  $\Delta$ .

So we have to prove the above claim. The shape of  $G$  implies the existence of an  $\alpha \in \Delta^1$  such that  $\alpha - e_i \in \Delta^2$  and the existence of an  $\alpha' \in \Delta^1$  such that  $\alpha' - e_j \in \Delta^2$ . Moreover,  $\alpha$  and  $\alpha'$  are connected by a sequence  $(\gamma_k)$  in  $\Delta^1$  such that step from each  $\gamma_k$  to  $\gamma_{k+1}$  is by a standard basis element, as in Definition 25. We may assume in addition that  $(\gamma_k)$  contains no loops. Given a fixed index  $k$ , one of the following assertions will hold:

- (1)  $\gamma_{k+1} = \gamma_k + e_i$ ,  $\gamma_{k+2} = \gamma_{k+1} + e_j$ ;
- (2)  $\gamma_{k+1} = \gamma_k + e_i$ ,  $\gamma_{k+2} = \gamma_{k+1} - e_j$ ;
- (3)  $\gamma_{k+1} = \gamma_k - e_i$ ,  $\gamma_{k+2} = \gamma_{k+1} + e_j$ ;
- (4)  $\gamma_{k+1} = \gamma_k - e_i$ ,  $\gamma_{k+2} = \gamma_{k+1} - e_j$ .

for some  $i \neq j$ . In each of these cases, we modify our sequence as follows:

- (1) if  $\gamma_k + e_j \in \Delta^1$ , we define  $\gamma_{k+1} := \gamma_k + e_j$ ;
- (2) if  $\gamma_k - e_j \in \Delta^1$ , we define  $\gamma_{k+1} := \gamma_k - e_j$ ;
- (3) if  $\gamma_k + e_j \in \Delta^1$ , we define  $\gamma_{k+1} := \gamma_k + e_j$ ;
- (4) if  $\gamma_k - e_j \in \Delta^1$ , we define  $\gamma_{k+1} := \gamma_k - e_j$ .

Doing so, we obtain another sequence  $(\gamma'_k)$  connecting  $\alpha$  and  $\alpha'$ . Upon iterating that process, modifying all sequences at all indices  $k$ , we obtain a set  $\Gamma$  of sequences connecting  $\alpha$  and  $\alpha'$ . We may assume that all members of sequences in  $\Gamma$  live in a large but finite hypercube  $\{0, \dots, e\}^{d-1} \subseteq \mathbb{N}^{d-1}$ . In particular,  $\Gamma$  itself is finite. So we will eventually find a sequence  $(\gamma_k)$  in  $\Delta^1$  such that for each member  $\gamma_k$ , there exists an  $i \in \{1, \dots, d-1\}$  such that the difference  $\gamma_k - e_i$  lies in  $\Delta^2$  or  $\Delta^4$ . Loosely speaking, the sequence  $(\gamma_k)$  sneaks along the border (in the sense of [1]) of the standard set  $\Delta^2 \cup \Delta^3 \cup \Delta^4$ .

Since the starting point of that sequence is  $\alpha$  and the endpoint is  $\alpha'$ , we find some index  $k$  such that  $\gamma_k - e_i \in \Delta^2$  and  $\gamma_{k+1} - e_j \in \Delta^4$ . We define

$$\beta := \begin{cases} \gamma_k & \text{if } \gamma_{k+1} = \gamma_k + e_l, \\ \gamma_{k+1} & \text{if } \gamma_{k+1} = \gamma_k - e_l. \end{cases}$$

In the first case,  $\beta - e_i \in \Delta^2$  and  $\beta + e_l - e_j \in \Delta^4$ , so also  $\beta - e_j \in \Delta^4$  by standardness of  $\Delta$ . The second case works similarly.  $\square$

The graphs in Figures 7 and 8 define relations on their respective node sets which both fail to be transitive. So one might not guess that transitivity of graphs in  $\mathcal{S}$  is crucial for such graphs to arise from standard sets. That, however, is indeed true, as we shall see in Proposition 30 below. Let us first establish that passing from a graph to its transitive closure has no impact on standard decompositions.

**Lemma 29.** Let  $G$  be a standard graph and  $\overline{G}$  its transitive closure. Then the standard decompositions of  $G$  and  $\overline{G}$  are in canonical bijection.

*Proof.* Given a standard decomposition  $\mathcal{H}$  of  $G$ , replace every member  $H$  by its transitive closure  $\overline{H}$ . The resulting multiset  $\overline{\mathcal{H}}$  is a standard decomposition of  $\overline{G}$ .

Given a standard decomposition  $\mathcal{K}$  of  $\overline{G}$ , we delete from every member  $K$  all edges that appear in  $\overline{G}$  but not in  $G$ , and call the resulting graph  $K^\circ$ . The resulting multiset  $\mathcal{K}^\circ$  is a standard decomposition of  $G$ . The maps  $\mathcal{H} \mapsto \overline{\mathcal{H}}$  and  $\mathcal{K} \mapsto \mathcal{K}^\circ$  are mutual inverses.  $\square$

**Proposition 30.** Let  $G$  be a connected and transitive standard graph containing a unique node of maximal label. Then there exists a standard set  $\Delta \subseteq \mathbb{N}^d$ , for some  $d \geq 1$ , whose canonicalized standard graph is  $G$ .

*Proof.* Using the terminology of Definition 25, we denote by  $G'(\Delta)$  the canonicalized standard graph of a standard set  $\Delta$ . We prove the proposition by two nested inductions, the outer over the number of nodes of  $G$ , and the inner over the number of edges of  $G$ . The base case of the outer induction is trivial. As for the outer induction step, let  $G$  be a given connected and transitive standard graph containing a unique node  $v_h$  of maximal label,  $h$ . We remove from  $G$  one node  $v_0$  of minimal label, along with all the edges whose source is  $v_0$ . We call the graph thus obtained  $G_0$ . Note that  $G_0$  is again transitive. As we do not change the labels of the remaining nodes, also  $G_0$  contains a unique node of maximal label. We may assume that there exists a standard set  $\Delta_0 \subseteq \mathbb{N}^d$ , for some  $d$ , such that  $G'(\Delta_0) = G_0$ .

For establishing the outer induction step, we shall put the node  $v_0$  back into the graph. Transitivity of  $G$  implies that this graph contains an edge from  $v_0$  to  $v_h$ . Let  $G_1$  be the (transitive) graph that arises from  $G_0$  by adding the one node  $v_0$  and the one edge  $(v_0, v_h)$ . We now construct a standard set  $\Delta_1$  such that  $G'(\Delta_1) = G_1$ .

Consider the embedding  $\iota : \mathbb{N}^d \hookrightarrow \mathbb{N}^{d+1} : \beta \mapsto (0, \beta)$ . The transition from  $\Delta_0$  to  $\iota(\Delta_0)$  does not affect the graph of  $\Delta_0$ . We may therefore assume that  $\Delta_0 \subseteq \mathbb{N}^d$  is contained in the hyperplane  $\{\beta_1 = 0\}$  of  $\mathbb{N}^d$ . The node  $v_h \in G_0$  corresponds to the isohypse  $(\Delta_0)^h$ . Let  $h_0 < h$  be the label of  $v_0$ . The set

$$\begin{aligned} \Delta_1 &:= \Delta_0 \cup M_1, \text{ where} \\ M_1 &:= \{(1, 0, \dots, 0, \beta_d) \mid 0 \leq \beta_d \leq h_0 - 1\} \end{aligned}$$

is standard. See Figure 9 for a visualization of the transition from  $\Delta_0$  to  $\Delta_1$ . For  $a \neq h_0$ , the isohypses  $(\Delta_0)^a$  and  $(\Delta_1)^a$  are identical. The isohypse  $(\Delta_1)^{h_0}$  is  $q^d(M_1) = \{(1, \dots, 0)\}$ . When passing to  $G'(\Delta_1)$ , we see that this graph arises from  $G'(\Delta_0)$  by adding the one node  $(\Delta_1)^{h_0}$  and the one edge connecting that new node and  $(\Delta_1)^h$ . This establishes the outer induction step, and at the same time the inner induction basis.

As for the inner induction step, we may assume to have a transitive graph  $G_1$

- with the same nodes and the same labels as  $G$ ,
- and a distinguished node  $v_0$
- such that all edges but those with source  $v_0$  agree in  $G$  and  $G_1$ ,

along with a standard set  $\Delta_1 \subseteq \mathbb{N}^d$  such that  $G'(\Delta_1) = G_1$ . Let  $v_1$  be a node of  $G$  such that  $(v_0, v_1)$  is an edge in  $G$ , but our original graph  $G$  contains no chain of edges from  $v_0$  to  $v_1$  of length more than 1. We may assume that  $\mathcal{L}_G(v_0) < \mathcal{L}_G(v_1)$ . Denote by  $G_2$  the graph that arises from  $G_1$  by adding the edge  $(v_0, v_1)$ . We will prove the existence of a standard set  $\Delta_2$  such that  $G'(\Delta_2) = G_2$ . This will establish the inner induction step, and finish the proof of the proposition.

Analogously as above, we assume that  $\Delta_1 \subseteq \mathbb{N}^d$  is contained in the hyperplane  $\{\beta_1 = 0\}$  of  $\mathbb{N}^d$ . The choice of  $v_1$  implies that  $G_2$  is again transitive. For  $j = 0, 1$ , the node  $v_j \in G_1$  corresponds to a connected component  $C_j$  of  $(\Delta_0)^{h_j}$ , where  $h_j$  is

the label of  $v_j$ . The set

$$\begin{aligned}\Delta_{1\frac{1}{2}} &:= \Delta_1 \cup M_{1\frac{1}{2}}, \text{ where} \\ M_{1\frac{1}{2}} &:= \left( \bigcup_{\alpha \in \mathbb{N}^d} ((q^d)^{-1}(C_1) \cap \Delta + e_1 - \alpha) \right) \cap \mathbb{N}^d\end{aligned}$$

is standard. See the first two pictures in Figure 10 for a visualization of the transition from  $\Delta_1$  to  $\Delta_{1\frac{1}{2}}$ : We create a copy of the set  $(q^d)^{-1}(C_1)$  in the hyperplane  $\{\beta_1 = 1\}$  of  $\mathbb{N}^d$  and subsequently pass to the smallest standard set containing both  $\Delta_1$  and that copy. Transitivity of  $G_1$  implies that  $G'(\Delta_{1\frac{1}{2}}) = G'(\Delta_1)$ . Indeed, for all heights  $a \neq h_1$ , the connected components of  $(\Delta_{1\frac{1}{2}})^a$  are identical to of the connected components of  $(\Delta_1)^a$ . For height  $h_1$ , the same is true for those connected components of  $(\Delta_{1\frac{1}{2}})^{h_1}$  that do not project to  $C_1$  under  $q^d$ . The connected component  $C_1$  of  $(\Delta_1)^{h_1}$ , however, has a much larger counterpart in  $\Delta_{1\frac{1}{2}}$ , namely, the union of  $C_1$  and the set  $q^d(M_{1\frac{1}{2}})$ . As for edges in  $G'(\Delta_{1\frac{1}{2}})$  emerging from node  $C_1 \cup q^d(M_{1\frac{1}{2}})$ , the presence of  $M_{1\frac{1}{2}}$  obviously leads to new adjacencies in connected components of isohypses of  $\Delta_{1\frac{1}{2}}$ . But transitivity of  $G_1$  guarantees that none of those adjacencies lead to an edge in  $G'(\Delta_{1\frac{1}{2}})$  that does exist in  $G'(\Delta_1)$ . So the graphs  $G'(\Delta_1)$  and  $G'(\Delta_{1\frac{1}{2}})$  are identical.

However, we do not want another standard set with the same canonicalized graph, but rather a graph with one additional edge. We obtain that edge by applying the same trick once more, defining

$$\begin{aligned}\Delta_2 &:= \Delta_1 \cup M_{1\frac{1}{2}} \cup M_2, \text{ where} \\ M_2 &:= \left( \bigcup_{\alpha \in \mathbb{N}^d} ((q^d)^{-1}(C_0) + e_1 - \alpha) \right) \cap \mathbb{N}^d.\end{aligned}$$

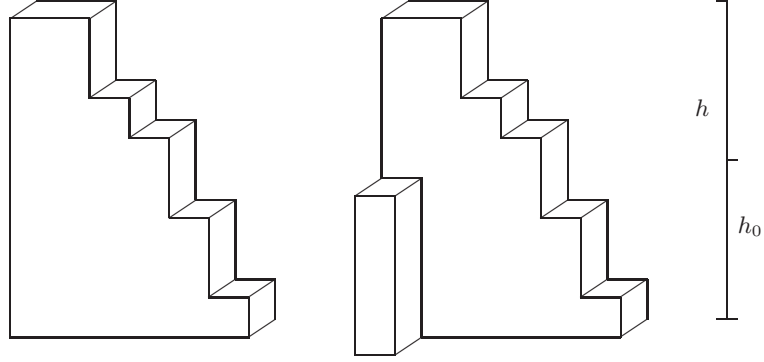
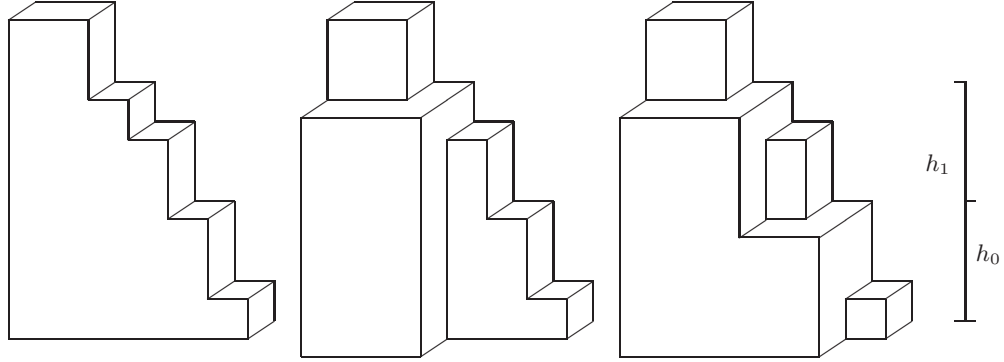
This is another standard set. See the last two pictures in Figure 10 for a visualization of the transition from  $\Delta_{1\frac{1}{2}}$  to  $\Delta_2$ : We also create a copy of the set  $(q^d)^{-1}(C_0)$  in the hyperplane  $\{\beta_1 = 1\}$  of  $\mathbb{N}^d$  and subsequently pass to the smallest standard set containing both  $\Delta_{1\frac{1}{2}}$  and that copy. For all heights  $a \neq h_0, h_1$ , the connected components of  $(\Delta_2)^a$  are identical to the connected components of  $(\Delta_{1\frac{1}{2}})^a$ . For heights  $a = h_0, h_1$ , the same is true for those connected components of  $(\Delta_2)^a$  that do not project to  $C_0$  or  $C_1$ . Note that the sets  $M_{1\frac{1}{2}}$  and  $M_2$  will in general intersect. The isohypse of  $C_0$ , however, does not contain any elements of  $M_2$ . The counterpart of  $C_1$  in  $\Delta_2$  is the union  $C_1 \cup M_{1\frac{1}{2}}$ ; and the counterpart of  $C_0$  in  $\Delta_2$  is  $(C_0 \cup M_2) \setminus M_{1\frac{1}{2}}$ . The graph  $G'(\Delta_2)$  contains all the edges that appear in  $G'(\Delta_{1\frac{1}{2}})$ , plus an edge from node  $(C_0 \cup M_2) \setminus M_{1\frac{1}{2}}$  to node  $C_1 \cup M_{1\frac{1}{2}}$ . This establishes the inner induction step.  $\square$

## 8. REDUCTION TO STANDARD GRAPHS WITH UNIQUE MAXIMAL NODES

Proposition 30 is the final step of three in proving that C4 decomposition and standard decomposition of labeled graphs are equivalent.

**Proposition 31.** Let  $G$  be a labeled graph. Then there exists a graph  $G'$  such that

- (1)  $G$  is a subgraph of  $G'$ ,


 FIGURE 9. From  $\Delta_0$  to  $\Delta_1$ 

 FIGURE 10. From  $\Delta_1$  to  $\Delta_{1\frac{1}{2}}$  and  $\Delta_2$ 

- (2)  $G'$  has a unique node of maximal label that is reachable from all nodes of  $G'$  and,
- (3) the standard decompositions of  $G$  and  $G'$  are in canonical bijection.

*Proof.* Let  $l$  be the maximal label among all nodes of  $G$ . Let  $G'$  be equal to  $G$  except that  $G'$  has an extra node  $v$  with label  $l + 1$  and  $v$  has an edge to it from all other nodes. The first two conditions are immediate, so it remains to show that the standard decompositions of  $G$  and  $G'$  are in canonical bijection.

Consider the function  $f$  defined by  $f(H) = f(H \setminus \{v\})$ . It is not hard to see that  $f$  is a bijection from the standard components of  $G'$  to the standard components of  $G$ . Extend  $f$  to map multisets of standard components by applying it to each standard component individually, so for example  $f(\{A, B\}) = \{f(A), f(B)\}$ .

Let  $D$  be a standard decomposition of  $G'$ . Write  $D$  as a union of  $D'$  and  $V$  where  $V$  is a multiset that contains only copies of  $\{v\}$  while  $D'$  does not contain  $\{v\}$ . Then it is not hard to see that  $f(D')$  is a decomposition of  $G$ .

For the other direction, let  $D$  be a standard decomposition of  $G$  and let  $D' := f^{-1}(D)$ . Then  $G' - \sum D'$  is a graph where all the nodes have label zero except

that  $v$  has a label  $l > 0$ . Let  $V$  be the multiset that contains  $l$  copies of  $\{v\}$ . Then  $D' \cup V$  is a decomposition of  $G'$ . Let  $g$  be the function  $D \mapsto D' \cup V$ . It is not hard to see that  $f$  and  $g$  are mutual inverses.  $\square$

We can now prove that C4 decomposition and standard decomposition of labeled graphs are equivalent.

**Theorem 32.** The two problems,

- (1) computing standard decompositions of labeled graphs and
- (2) computing C4 decompositions of finite standard sets

are equivalent.

*Proof of Theorem 32.* A solution of problem (1) implies a solution of problem (2) by Proposition 24. Assume we are able to solve problem (2), and are given a labeled graph  $G$ . We erase from  $G$  all parallel edges and loops and pass to the canonicalization  $G'$ . This graph has the same standard decompositions as  $G$ . Moreover, since the function that computes all standard decompositions of a graph turns coproducts into products, we may consider each connected component of  $G'$  individually. If a component has multiple nodes of locally maximal label, we pass to a graph  $G''$  with only one node of maximal (and sufficiently large) label, as in Proposition 31. The cited proposition says that this transition does not harm the decompositions. Then we replace  $G''$  by its transitive closure  $G'''$ , which transition does not harm the decompositions either, see Lemma 29. Finally, Proposition 30 provides a standard set  $\Delta'''$  whose canonicalized standard graph is  $G'''$ . Problem (1) is solved.  $\square$

## APPENDIX A. A GENERATING FUNCTION

We will now present a natural generating function for the number of standard decompositions of a standard graph  $G$ . We leave the translation of the graph setting into the standard sets setting to interested readers.

It is good to temporarily forget about labelings. So let  $F$  be an unlabeled directed graph. Let  $\mathcal{E}$  be the set of all standard 0-1 subgraphs of  $F^4$  with node set  $\mathcal{V}_F$ . We identify each  $E \in \mathcal{E}$  with the *characteristic function* of the labeling, that is, with the vector  $\chi_E := (\chi_{E,v})_{v \in \mathcal{V}_F}$  indexed by nodes of  $F$ , with entries

$$\chi_{E,v} := \begin{cases} 1 & \text{if } v \in \mathcal{V}_E \\ 0 & \text{else.} \end{cases}$$

We define  $\chi := (\chi_{E,v})_{E \in \mathcal{E}, v \in \mathcal{V}_F}$  to be the matrix whose rows are indexed by  $\mathcal{E}$ , the row with index  $E$  being the vector  $\chi_E$ . Moreover, we introduce a vector  $t := (t_v)_{v \in \mathcal{V}_F}$  of indeterminates, also indexed by nodes of  $F$ . If  $w := (w_v)_{v \in \mathcal{V}_F}$  is any vector of nonnegative integers, indexed by nodes of  $F$ , we write  $t^w := \prod_{v \in \mathcal{V}_F} t_v^{w_v}$ . Consider the power series

$$g := \prod_{E \in \mathcal{E}} \frac{1}{1 - t^{\chi_E}}.$$

---

<sup>4</sup>We defined standard 0-1 subgraphs only for labeled graphs; if  $F$  is unlabeled, we give each node the trivial label 1; then the notion of 0-1 subgraphs is well-defined

We define integers  $\Phi_\chi(w)$ , one for each integer-valued vector  $w$  as above, by expanding the power series  $g$ ,

$$g =: \sum_{v \in \mathbb{N}^{\mathcal{V}_F}} \Phi_\chi(w) t^w.$$

$\Phi_\chi$  is called a *vector partition function*, see [4]. Note that labelings of graphs  $G$  with the same nodes and edges as  $F$  correspond to vectors  $w$  as above via

$$w = (\mathcal{L}_G(v))_{v \in \mathcal{V}_F}.$$

We denote by  $G_w$  the labeled graph  $G$  with the same nodes and edges as  $F$  and labeling given by  $w$ .

**Proposition 33.** (1) Given any vector  $w \in \mathbb{N}^{\mathcal{V}_F}$ , the coefficient  $\Phi_\chi(w)$  vanishes unless the labeled graph  $G_w$  is standard.  
 (2) If the labeled graph  $G_w$  is standard, the coefficient  $\Phi_\chi(w)$  equals the number of standard decompositions of  $F$ .

*Proof.* We expand each term  $\frac{1}{1-t^{\chi_E}}$  in the product expression of  $g$  as a geometric series,

$$g = \prod_{E \in \mathcal{E}} (1 + t^{\chi_E} + t^{2 \cdot \chi_E} + t^{3 \cdot \chi_E} + t^{4 \cdot \chi_E} + \dots).$$

Upon expanding the product, we see that each monomial appearing in the series takes the shape  $m = \prod_{E \in \mathcal{F}} t^{n_E \cdot \chi_E}$  for some finite  $\mathcal{F} \subseteq \mathcal{E}$  and some  $n_E \in \mathbb{N}$ . We replace the set  $\mathcal{F}$  by the multiset  $\mathcal{H}$  in which each  $E \in \mathcal{F}$  appears  $n_E$  times. Since each member of  $\mathcal{H}$  is a standard 0-1 subgraph of  $F$ , the graph  $G := \sum \mathcal{H}$  is standard and has the same nodes and edges as  $F$ . The above monomial  $m$  equals  $\prod_{v \in \mathcal{V}_F} t^{\mathcal{L}_G(v)}$ . This establishes (1).

As for (2), let  $G$  be a standard graph with the same nodes and edges as  $\mathcal{V}_F$ . The above discussion shows that the coefficient of the monomial  $m := \prod_{v \in \mathcal{V}_F} t^{\mathcal{L}_G(v)}$  shows up in the expansion of  $g$ , and its coefficient counts the number of ways of writing  $G$  as a sum  $G = \sum \mathcal{H}$  of elements of  $\mathcal{E}$ . This is just the number of standard decompositions of  $G$ .  $\square$

## APPENDIX B. PARTITIONS OF PARTITIONS

Appendix A suggests a connection between standard decompositions and partitions. Let us further investigate this.

**Example 34.** • The set of partitions of an integer  $n$  is in natural bijection with the set of standard sets of cardinality  $n$  by identifying a partition and its Young diagram (in the French notation).  
 • If  $p = \{n_1, \dots, n_h\}$  (a multiset) is a partition of  $n$  and for each  $i$ ,  $p_i$  is a partition of  $n_i$ , we call  $\{p_1, \dots, p_h\}$  a *partition of partition of  $n$* . The set of partitions of partitions of  $n$  is in natural bijection with the set of standard sets  $\Delta \subseteq \mathbb{N}^3$  of cardinality  $n$ , together with all their  $C_4$  decompositions.

Both bijections are visualized in Figure 11. Here are the general statements. The crucial definition is that of a *C<sub>4</sub> game*. Note that a C<sub>4</sub> game in  $\mathbb{N}^2$  is the same thing as a standard set in  $\mathbb{N}^2$  since each standard set in  $\mathbb{N}^2$  has a unique C<sub>4</sub> decomposition.

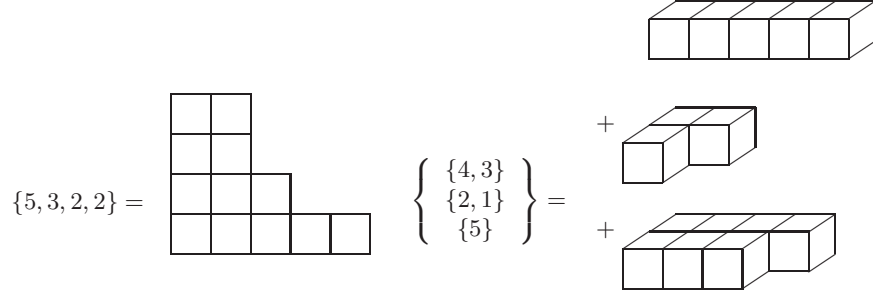


FIGURE 11. Partitions (of partitions, resp.) and C4 games in  $\mathbb{N}^2$  (in  $\mathbb{N}^3$ , resp.) correspond to each other

**Definition 35** (Iterated partition). Let  $n$  be a positive integer. We recursively define a  $q$ -fold iterated partition of  $n$  as follows:

- for  $q = 1$ , it is a partition of  $n$ , that is, a multiset  $p = \{n_1, \dots, n_h\}$  of positive integers such that  $\sum n_i = n$ ;
- for  $q > 1$ , it is a multiset  $p = \{p_1, \dots, p_h\}$  of  $(q - 1)$ -fold iterated partitions of integers  $n_1, \dots, n_h$  such that  $\sum n_i = n$ .

In other words, we look at all partitions of  $n$  into  $n_i$ , together with all partitions of all parts  $n_i$  into  $n_{i,j}$ , together with all partitions of all parts  $n_{i,j}$  into  $n_{i,j,k}$ , etc.

**Definition 36** (C4 game). Let  $n$  be a positive integer. We recursively define a C4 game of size  $n$  in  $\mathbb{N}^d$  as follows:

- for  $d = 1, 2$ , it is standard set  $\Delta \subseteq \mathbb{N}^d$  of cardinality  $n$ ;
- so for  $d = 2$ , we may identify  $\Delta \subseteq \mathbb{N}^2$  with a multiset  $\{\Delta_1, \dots, \Delta_h\}$  of standard sets  $\Delta_i \subseteq \mathbb{N}$  such that their C4 sum equals  $\Delta$ ;
- for  $d > 2$ , it is a multiset  $\{g_1, \dots, g_h\}$  of C4 games of respective sizes  $n_i$  in  $\mathbb{N}^{d-1}$  such that  $\sum n_i = n$ .

In other words, we look at all standard sets  $\Delta \subseteq \mathbb{N}^d$  of a cardinality  $n$ , together with all C4 decompositions of  $\Delta$  into  $\Delta_i \subseteq \mathbb{N}^{d-1}$ , together with all C4 decompositions of all  $\Delta_i$  into  $\Delta_{i,j} \subseteq \mathbb{N}^{d-2}$ , together with all C4 decompositions of all  $\Delta_{i,j}$  into  $\Delta_{i,j,k} \subseteq \mathbb{N}^{d-3}$ , etc.

**Proposition 37.** For all  $d, n \in \mathbb{N}$ , there is a natural bijection

$$f_d : \{(d - 1)\text{-fold iterated partitions of } n\} \rightarrow \{\text{C4 games of size } n \text{ in } \mathbb{N}^d\}.$$

*Proof.* The assertion is obvious for  $d = 1, 2$ . For  $d > 2$ , the bijection  $f_d$  sends each multiset  $\{H_1, \dots, H_l\}$  of  $(d - 2)$ -fold iterated partitions of integers  $n_1, \dots, n_l$  to the multiset  $\{f_{d-1}(H_1), \dots, f_{d-1}(H_l)\}$ .  $\square$

Note that the bijection is only natural up to the choice of coordinate axes in  $\mathbb{N}^d$ . In other words, replacing the tuple  $(e_1, \dots, e_d)$  of standard basis elements by  $(e_{\sigma(1)}, \dots, e_{\sigma(d)})$  for some permutation  $\sigma$  induces an automorphism of the source of bijection  $f_d$ . For  $d = 2$ , this corresponds to the ambiguity between a partition and its transpose.

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